Lagrange Multiplier (Part I)

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Sometimes we need to maximize or minimize a function under certain constrains. For example, we want to maximize the area of a rectangle given that the perimeter is fixed. This type of question is called optimisation with constraints. One way you may come up with to solve this type of problem is to express one variable in terms of the others, and reduce the problem into one with one less variables. This idea works sometimes, but in general, it may be difficult to express one variable explicitly as a formula of the others. We therefore need to develop a more general method.

Definition 1. $f(x, y)$ and $g(x, y)$ are functions and c is a real number. The question of finding maximum or minimum of f on the level set $g(x, y) = c$ is called optimisation with constraints. The level set $g(x, y) = c$ is called the constraint.

Theorem 2. *(Method of Lagrange Multiplier) To find the maximum and minimum values of $f(x, y)$ subject to the constraint $g(x, y) = c$ (assuming that extreme values exist, $\nabla g(x, y) \neq \vec{0}$ on $g(x, y) = c$):*

(i). Solve for the system of equations

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = c \end{cases} \tag{0.1}$$

The number λ is called the Lagrange multiplier

(ii). The extreme points are among the solutions in (i).

Example 3. *Maximize the function $f(x, y) = x + y$ on the unit circle $x^2 + y^2 = 1$.*

Let $g(x,y) = x^2 + y^2$, the question is to maximize $f(x,y)$ subject to the constraint $g(x,y) = 1$. By the method of Lagrange multiplier, we get

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 1 \end{cases}$$

i.e.

$$\begin{cases} (1,1) = \lambda \nabla (2x, 2y) = (2\lambda x, 2\lambda y) \\ x^2 + y^2 = 1 \end{cases}$$

i.e.

$$\begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

Solving this system of equations, we get $\lambda = x = y = \frac{\sqrt{2}}{2}$ or $\lambda = x = y = -\frac{\sqrt{2}}{2}$
$f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = \sqrt{2}$ and $f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = -\sqrt{2}$, so f attains maximum at $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ with maximum value $\sqrt{2}$ under the constrain.

Example 4. *A person has utility function $u(x,y) = 10xy + 5x + 2y$. Suppose the price for one unit of x is 2 dollars and the price for one unit of y is 5 dollars. If the person has 100 dollars that can be spent on x and y, find x and y that maximize the utility.*

The question is to maximize $u(x,y) = 10xy + 5x + 2y$ under the constraint $2x + 5y = 100$.

Let $g(x,y) = 2x + 5y$, by the Lagrange method, we have

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = 100 \end{cases}$$

i.e.

$$\begin{cases} 10y + 5 = \lambda \times 2 \\ 10x + 2 = \lambda \times 5 \\ 2x + 5y = 100 \end{cases}$$

We get $\lambda = 51.45, x = 25.525, y = 9.79$.

The method of Lagrange multiplier can be generalized to more variables and more constraints:

Theorem 5. *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = c$ (assuming that extreme values exist, $\nabla g(x, y, z) \neq \vec{0}$ on $g(x, y, z) = c$):*

(i). Solve for the system of equations

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = c \end{cases} \tag{0.2}$$

The number λ is called the Lagrange multiplier

(ii). The extreme points are among the solutions in (i).

Example 6. *A rectangular box without lid is to be made from 12 square metres of cardboard. Find the maximum volume of such a box.*

Let the length, width and height of the box be x, y, z. The volume is $V(x, y, z) = xyz$. The constraint is the area of surface $A(x, y, z) = 2xz + 2yz + xy = 12$.

By the method of Lagrange multiplier:

$$\begin{cases} \nabla V(x, y, z) = \lambda \nabla A(x, y, z) \\ A(x, y, z) = c \end{cases}$$

which becomes

$$\begin{cases} yz = \lambda(2z + y) \\ xz = \lambda(2z + x) \\ xy = \lambda(2x + 2y) \\ 2xz + 2yz + xy = 12 \end{cases}$$

Solving the system of equations, we get $x = 2, y = 2, z = 1, \lambda = \frac{1}{2}$. We conclude the maximum volume is obtained at $x = 2, y = 2, z = 1$, and the maximum volume is $2 \times 2 \times 1 = 4$.

Sometimes we also need to deal with optimisation subject to two constraints:

Theorem 7. *To find the maximum and minimum values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = c$ and $h(x, y, z) = k$(assuming that extreme values exist, and $\nabla g(x, y, z)$ is not parallel to $\nabla h(x, y, z)$):*

(i). Solve for the system of equations

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = c \\ h(x,y,z) = k \end{cases} \tag{0.3}$$

The numbers λ, μ are called the Lagrange multipliers

(ii). The maximum and minimum points are among the solutions in (i).

Example 8. *Find the maximum value of $f(x,y,z) = x + 2y + 3z$ subject to the constraints $x - y + z = 1$ and $x^2 + y^2 = 1$.*

Let $g(x,y,z) = x - y + z = 1$, $h(x,y,z) = x^2 + y^2 = 1$. By the method of Lagrange multiplier,

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = x - y + z = 1 \\ h(x,y,z) = x^2 + y^2 = 1 \end{cases}$$

i.e.

$$\begin{cases} (1,2,3) = \lambda(1,-1,1) + \mu(2x, 2y, 0) \\ g(x,y,z) = x - y + z = 1 \\ h(x,y,z) = x^2 + y^2 = 1 \end{cases}$$

i.e.

Solving the system of equations, we get $x = -\frac{2}{\sqrt{29}}, y = \frac{5}{\sqrt{29}}, z = 1 + \frac{7}{\sqrt{29}}, \lambda = 3, \mu = \frac{\sqrt{29}}{2}$ or $x = \frac{2}{\sqrt{29}}, y = -\frac{5}{\sqrt{29}}, z = 1 - \frac{7}{\sqrt{29}}, \lambda = 3, \mu = -\frac{\sqrt{29}}{2}$

$f(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}) = 3 + \sqrt{29}$ and $f(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}) = 3 - \sqrt{29}$

So the maximum value is $3 + \sqrt{29}$.

We will explain the reason why the method of Lagrange Multiplier works and we will also try to understand the meaning of Lagrange multiplier λ in the next section.